

# OBSTACLE PROBLEM FOR SPDE WITH NONLINEAR NEUMANN BOUNDARY CONDITION VIA REFLECTED GENERALIZED BACKWARD DOUBLY SDES

AUGUSTE AMAN<sup>\*†</sup>

U.F.R.M.I, Université de Cocody,  
582 Abidjan 22, Côte d'Ivoire

N. MRHARDY<sup>‡</sup>

F.S.S.M, Université Cadi Ayyad,  
2390, Marrakech, Maroc

## Abstract

This paper is intended to give a representation for stochastic viscosity solution of semi-linear reflected stochastic partial differential equations with nonlinear Neumann boundary condition. We use its connection with reflected generalized backward doubly stochastic differential equations.

**AMS Subject Classification:** 60H15; 60H20

**Keywords:** Backward doubly SDEs, Stochastic PDEs, Obstacle problem, stochastic viscosity solutions.

## 1 Introduction

Backward stochastic differential equations (BSDEs, for short) were introduced by Pardoux and Peng [12] in 1990, and it was shown in various papers that stochastic differential equations (SDEs) of this type give a probabilistic representation for solution (at least in the viscosity sense) of a large class of system of semi-linear parabolic partial differential equations (PDEs). Thereafter a new class of BSDEs, called backward doubly stochastic (BDSDEs), was considered by Pardoux and Peng [13]. It seems suitable for giving a representation for a system of parabolic stochastic partial differential equations (SPDEs). We refer to Pardoux and Peng [13] for the link between SPDEs and BDSDEs when the solutions of SPDEs are regular i.e the coefficients are smooth enough (at least in  $C^3$ ). The more general situation is much more delicate to treat because of the difficulties of extending the notion of viscosity solutions to SPDEs.

---

<sup>\*</sup>Supported by AUF post doctoral grant 07-08, Réf:PC-420/2460

<sup>†</sup>augusteam5@yahoo.fr, Corresponding author.

<sup>‡</sup>n.mrhardy@ucam.ac.ma

The notion of viscosity solution for PDEs was introduced by Crandall, Ishii and Lions [5] for certain first-order Hamilton-Jacobi equations. Today this theory becomes an important tool in many applied fields, especially in optimal control theory and numerous subjects related to it.

The stochastic viscosity solution for semi-linear SPDEs was introduced firstly by Lions and Souganidis in [8]. They use the so-called "stochastic characteristic" to remove the stochastic integrals from a SPDEs. A few years later, two others definitions of stochastic viscosity solution of SPDEs are considered by Buckdahn and Ma respectively in [2, 3] and [4]. In [2, 3], they used the "Doss-Sussman" transformation to connect the stochastic viscosity solution of SPDEs with the solution of associated BDSDEs. In [4], they introduced the stochastic viscosity solution by using the notion of stochastic sub and super jets. Next, in order to give a representation for viscosity solution of SPDEs with nonlinear Neumann boundary condition, Boufoussi et al. [1] introduced the so-called generalized BDSDEs. They refer the first technique (Doss-Sussman transformation) of Buckdahn and Ma [2, 3].

Motivated by the work of Boufoussi et al. [1] and employing the penalization method, we aim to establish the existence of viscosity solution for semi-linear reflected SPDEs with nonlinear Neumann boundary condition of the form:

$$O\mathcal{P}^{(f,\phi,g,h,l)} \left\{ \begin{array}{l} (i) \min \left\{ u(t,x) - h(t,x), -\frac{\partial u(t,x)}{\partial t} - [Lu(t,x) + f(t,x,u(t,x), \sigma^*(x)D_x u(t,x))] \right. \\ \left. - g(t,x,u(t,x))\dot{B}_s \right\} = 0, \quad (t,x) \in [0,T] \times \Theta \\ (ii) \frac{\partial u}{\partial n}(t,x) + \phi(t,x,u(t,x)) = 0, \quad (t,x) \in [0,T] \times \partial\Theta, \\ (iii) u(T,x) = l(x), \quad x \in \bar{\Theta} \end{array} \right.$$

where  $\dot{B}$  denotes white noise,  $L$  is the infinitesimal generator of a diffusion process  $X$ ,  $\Theta$  is a connected bounded domain included in  $\mathbb{R}^d$ , ( $d \geq 1$ ); and  $f, g, \phi, l, h$  are some measurable functions.

More precisely, we give a direct links with the following reflected generalized BDSDE:

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T \phi(s, Y_s) dA_s + \int_t^T g(s, Y_s) \overleftarrow{dB}_s \\ & - \int_t^T Z_s dW_s + K_T - K_t, \quad 0 \leq t \leq T, \end{aligned}$$

where  $\xi$  is the terminal value,  $A$  is a positive real-valued increasing process and  $dW$  and  $\overleftarrow{dB}$  denote respectively the classical forward and backward Itô integral. Our work generalize [15] in where authors treat deterministic reflected PDEs with nonlinear Neumann boundary conditions i.e  $g \equiv 0$  and the second appears in [1] where the non reflected SPDE with nonlinear Neumann boundary condition is considered.

The present paper is organized as follows: An existence and uniqueness result to large class of reflected generalized BDSDEs is shown in Section 2. Section 3 is devoted to give a definition of a reflected stochastic solution to SPDEs and establishes its existence result.

## 2 Reflected generalized backward doubly stochastic differential equations

### 2.1 Notation, assumptions and definition.

The scalar product of the space  $\mathbb{R}^d (d \geq 2)$  will be denoted by  $\langle \cdot, \cdot \rangle$  and the associated Euclidian norm by  $\|\cdot\|$ .

In what follows let us fix a positive real number  $T > 0$ . First of all  $\{W_t, 0 \leq t \leq T\}$  and  $\{B_t, 0 \leq t \leq T\}$  are two mutually independent standard Brownian motions with values respectively in  $\mathbb{R}^d$  and  $\mathbb{R}^\ell$ , defined respectively on the two complete probability spaces  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $(\Omega', \mathcal{F}', \mathbb{P}')$ . Let  $\mathbf{F}^B = \{\mathcal{F}_{t,T}^B\}_{t \geq 0}$  denote a retrograde filtration generated by  $B$ , augmented by the  $\mathbb{P}$ -null sets of  $\mathcal{F}$ ; and let  $\mathcal{F}_T^B = \mathcal{F}_{0,T}^B$ . We also consider the following family of  $\sigma$ -fields:

$$\mathcal{F}_t^W = \sigma\{W_s, 0 \leq s \leq t\}.$$

Next we consider the product space  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$  where

$$\overline{\Omega} = \Omega \times \Omega', \quad \overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}' \quad \text{and} \quad \overline{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'.$$

For each  $t \in [0, T]$ , we define

$$\mathcal{F}_t = \mathcal{F}_{t,T}^B \otimes \mathcal{F}_t^W \vee \overline{\mathcal{N}} \quad \text{and} \quad \mathcal{G}_t = \mathcal{F}_T^B \otimes \mathcal{F}_t^W \vee \overline{\mathcal{N}}.$$

where  $\overline{\mathcal{N}}$  denotes all the  $\overline{\mathbb{P}}$ -null sets in  $\overline{\mathcal{F}}$ .

The collection  $\mathbf{F} = \{\mathcal{F}_t, t \in [0, T]\}$  is neither increasing nor decreasing and it does not constitute a filtration. However,  $(\mathcal{G}_t)$  is a filtration.

Further, we assume that random variables  $\xi(\omega)$ ,  $\omega \in \Omega$  and  $\zeta(\omega')$ ,  $\omega' \in \Omega'$  are considered as random variables on  $\overline{\Omega}$  via the following identification:

$$\xi(\omega, \omega') = \xi(\omega); \quad \zeta(\omega, \omega') = \zeta(\omega').$$

In the sequel, let  $\{A_t, 0 \leq t \leq T\}$  be a continuous, increasing and  $\mathbf{F}$ -adapted real valued process such that  $A_0 = 0$ .

For any  $d \geq 1$ , we consider the following spaces of processes:

1.  $M^2(0, T, \mathbb{R}^d)$  denote the Banach space of all equivalence classes (with respect to the measure  $d\mathbb{P} \otimes dt$ ) where each equivalence class contains an  $d$ -dimensional jointly measurable stochastic process  $\phi_t; t \in [0, T]$ , such that: for all  $\mu > 0$

$$(i) \quad \|\phi\|_{M^2}^2 = \mathbb{E} \int_0^T e^{\mu A_t} |\phi_t|^2 dt < \infty;$$

$$(ii) \quad \phi_t \text{ is } \mathcal{F}_t\text{-measurable, for any } t \in [0, T].$$

2.  $S^2([0, T], \mathbb{R})$  is the set of one dimensional continuous stochastic processes which verify: for all  $\mu$

$$(iii) \quad \|\phi\|_{S^2}^2 = \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\mu A_t} |\phi_t|^2 + \int_0^T e^{\mu A_s} |\phi_s|^2 dA_s \right) < \infty;$$

(iv)  $\phi_t$  is  $\mathcal{F}_t$ -measurable, for any  $t \in [0, T]$ .

In addition, we give the following assumptions on the data  $(\xi, f, g, \phi, S)$ :

**(H1)**  $\xi$  is an  $\mathcal{F}_T$ -measurable, square integrable random variable; such that for all  $\mu > 0$

$$\mathbb{E} \left( e^{\mu A_T} |\xi|^2 \right) < \infty.$$

**(H2)**  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^\ell$ , and  $\phi : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , are three functions verifying:

(a) There exist  $\mathcal{F}_t$ -measurable processes  $\{f_t, \phi_t, g_t : 0 \leq t \leq T\}$  with values in  $[1, +\infty)$  such that for any  $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ , and any  $\mu > 0$ , the following hypotheses are satisfied for  $K > 0$ :

$$\left\{ \begin{array}{l} f(t, y, z), \phi(t, y), \text{ and } g(t, y, z) \text{ are } \mathcal{F}_t\text{-measurable processes,} \\ |f(t, y, z)| \leq f_t + K(|y| + \|z\|), \\ |\phi(t, y)| \leq \phi_t + K|y|, \\ |g(t, y, z)| \leq g_t + K(|y| + \|z\|), \\ \mathbb{E} \left( \int_0^T e^{\mu A_t} f_t^2 dt + \int_0^T e^{\mu A_t} g_t^2 dt + \int_0^T e^{\mu A_t} \phi_t^2 dA_t \right) < \infty. \end{array} \right.$$

(b) There exist constants  $c > 0$ ,  $\beta < 0$  and  $0 < \alpha < 1$  such that for any  $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$ ,

$$\left\{ \begin{array}{l} (i) |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2), \\ (ii) |g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \leq c|y_1 - y_2|^2 + \alpha \|z_1 - z_2\|^2, \\ (iii) \langle y_1 - y_2, \phi(t, y_1) - \phi(t, y_2) \rangle \leq \beta |y_1 - y_2|^2. \end{array} \right.$$

**(H3)** The obstacle  $\{S_t, 0 \leq t \leq T\}$ , is a continuous,  $\mathcal{F}_t$ -measurable, real-valued process, satisfying for any  $\mu > 0$

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\mu A_t} |S_t^+|^2 \right) < \infty,$$

and  $S_T \leq \xi$  a.s.

One of our main goals in this paper is the study of reflected generalized BDSDEs,

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T \phi(s, Y_s) dA_s + \int_t^T g(s, Y_s, Z_s) \overleftarrow{dB}_s \\ &\quad - \int_t^T Z_s dW_s + K_T - K_t, \quad 0 \leq t \leq T. \end{aligned} \tag{2.1}$$

First of all let us give a definition of the solution of this BDSDEs.

**Definition 2.1.** By a solution of the reflected generalized BDSDE  $(\xi, f, \phi, g, S)$  we mean a triplet of processes  $(Y, Z, K)$ , which satisfies (2.1) such that the following holds  $\mathbb{P}$ - a.s

(i)  $(Y, Z) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$

(ii) the map  $s \mapsto Y_s$  is continuous

(iii)  $Y_t \geq S_t, \quad 0 \leq t \leq T,$

(iv)  $K$  is an increasing process such that  $K_0 = 0$  and  $\int_0^T (Y_t - S_t) dK_t = 0.$

In the sequel,  $C$  denotes a positive constant which may vary from one line the other.

## 2.2 Comparison theorem

Let us give this needed comparison theorem related to the generalized BDSDE associated to  $(\xi, f, \phi, g)$  in the form

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T \phi(s, Y_s) dA_s + \int_t^T g(s, Y_s, Z_s) \overleftarrow{dB}_s \\ & - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \end{aligned}$$

It's proof follows with the same computation as in [17], with slight modification due to the presence of the integral with respect the increasing process  $A$ . So we just repeat the main step.

**Theorem 2.1.** (*Comparison theorem for generalized BDSDE*) Let  $(Y, Z)$  and  $(Y', Z')$  be the unique solution of the non reflected generalized BDSDE associated to  $(\xi, f, \phi, g)$  and  $(\xi', f', \phi', g)$  respectively. If  $\xi \leq \xi', f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$  and  $\phi(t, Y'_t) \leq \phi'(t, Y'_t)$ , then  $Y_t \leq Y'_t, \forall t \in [0, T]$ .

*Proof.* Let us set  $\Delta Y = Y - Y', \Delta Z = Z - Z'$  and  $(\Delta Y)^+ = (Y - Y')^+$  (with  $f^+ = \sup\{f, 0\}$ ). Using Itô's formula, we get, for all  $0 \leq t \leq T$

$$\begin{aligned} & \mathbb{E}((\Delta Y_t)^+)^2 + \mathbb{E} \int_t^T \|\Delta Z_s\|^2 \mathbf{1}_{\{Y_s > Y'_s\}} ds \\ & \leq \mathbb{E}((\xi - \xi')^+)^2 + 2\mathbb{E} \int_t^T (\Delta Y_s)^+ \mathbf{1}_{\{Y_s > Y'_s\}} \{f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)\} ds \\ & \quad + 2\mathbb{E} \int_t^T (\Delta Y_s)^+ \mathbf{1}_{\{Y_s > Y'_s\}} \{\phi(s, Y_s) - \phi'(s, Y'_s)\} dA_s \\ & \quad + \mathbb{E} \int_t^T \|g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)\|^2 \mathbf{1}_{\{Y_s > Y'_s\}} ds, \end{aligned} \tag{2.2}$$

where  $\mathbf{1}_\Gamma$  denotes the characteristic function of a given set  $\Gamma \in \mathbf{F}$  defined by

$$\mathbf{1}_\Gamma(\omega) = \begin{cases} 1 & \text{if } \omega \in \Gamma, \\ 0 & \text{if } \omega \notin \Gamma. \end{cases}$$

From **(H2)(b)**, we have

$$\begin{aligned} 2(\Delta Y_s)^+ \{f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)\} & \leq 2(\Delta Y_s)^+ \{f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)\} \\ & \leq \left(\frac{1}{\varepsilon} + 2c\right)((\Delta Y_s)^+)^2 + \varepsilon c \|\Delta Z_s\|^2, \end{aligned}$$

$$\begin{aligned} 2(\Delta Y_s)^+ \{ \phi(s, Y_s) - \phi(s, Y'_s) \} &\leq 2(\Delta Y_s)^+ \{ \phi(s, Y_s) - \phi(s, Y'_s) \} \\ &\leq 2\beta((\Delta Y_s)^+)^2 \end{aligned}$$

and

$$\|g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)\|^2 \mathbf{1}_{\{Y_s > Y'_s\}} \leq c((\Delta Y_s)^+)^2 \mathbf{1}_{\{Y_s > Y'_s\}} + \alpha \|\Delta Z_s\|^2 \mathbf{1}_{\{Y_s > Y'_s\}}.$$

Plugging these inequalities in (2.2) and choosing  $\varepsilon = \frac{1-\alpha}{2c}$ , we conclude that

$$\mathbb{E}((\Delta Y_t)^+)^2 \leq 0$$

which leads to  $\Delta Y_t^+ = 0$  a.s. and so  $Y'_t \geq Y_t$  a.s. for all  $t \leq T$ .  $\square$

### 2.3 Existence and uniqueness result

Our main goal in this section is to prove the following theorem.

**Theorem 2.2.** *Under the hypotheses (H1)-(H3), the reflected generalized BDSDE (2.1) has a unique solution  $(Y, Z, K)$ .*

Before we start proving this theorem, let us establish the same result in case  $g$  do not depends on  $Y$  and  $Z$ . More precisely, given  $g$  such that

$$\mathbb{E} \left( \int_0^T e^{\mu A_s} \|g(s)\|^2 ds \right) < \infty$$

and  $f$ ,  $\phi$  and  $\xi$  as above, consider the reflected generalized BDSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T \phi(s, Y_s) dA_s + \int_t^T g(s) \overleftarrow{dB}_s - \int_t^T Z_s dW_s + K_T - K_t. \quad (2.3)$$

**Proposition 2.1.** *There exists a unique triplet  $(Y, Z, K)$  verifies conditions (i)-(iv) of definition 2.1 and satisfies (2.3).*

*Proof.* **Existence**

It is based on a slight adaptation of the penalization method taking account the presence of the backward Itô integral with respect the Brownian motion  $B$ . For each  $n \in \mathbb{N}^*$ , we set

$$f_n(s, y, z) = f(s, y, z) + n(y - S_s)^- \quad (2.4)$$

and consider the generalized BDSDE

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds + \int_t^T \phi(s, Y_s^n) dA_s \\ &\quad + \int_t^T g(s) \overleftarrow{dB}_s - \int_t^T Z_s^n dW_s. \end{aligned} \quad (2.5)$$

It is well known (see Boufoussi et al., [1]) that generalized BDSDE (2.5) has a unique solution  $(Y^n, Z^n) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$  such that for each  $n \in \mathbb{N}^*$ ,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\mu A_t} |Y_t^n|^2 + \int_0^T e^{\mu A_s} \|Z_s^n\|^2 ds \right) < \infty.$$

On the other hand, for all  $n \geq 0$  and  $s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ ,

$$f_n(s, y, z) \leq f_{n+1}(s, y, z),$$

which provide by Theorem 2.1,  $Y_t^n \leq Y_t^{n+1}$ ,  $t \in [0, T]$  a.s. Therefore, setting  $Y_t = \sup_n Y_t^n$  we have

$$Y_t^n \nearrow Y_t \text{ a.s.}$$

*Step 1: A priori estimate*

For any  $\mu > 0$ , there exists  $C > 0$  such that,

$$\sup_{n \in \mathbb{N}^*} \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\mu A_t} |Y_t^n|^2 + \int_0^T e^{\mu A_s} |Y_s^n|^2 dA_s + \int_0^T e^{\mu A_s} \|Z_s^n\|^2 ds + |K_T^n|^2 \right) < C$$

where

$$K_t^n = n \int_0^t (Y_s^n - S_s)^- ds, \quad 0 \leq t \leq T. \quad (2.6)$$

Indeed, from Itô's formula it follows that

$$\begin{aligned} & e^{\mu A_t} |Y_t^n|^2 + \int_t^T e^{\mu A_s} \|Z_s^n\|^2 ds \\ & \leq e^{\mu A_T} |\xi|^2 + 2 \int_t^T e^{\mu A_s} Y_s^n f(s, Y_s^n, Z_s^n) ds + 2 \int_t^T e^{\mu A_s} Y_s^n \phi(s, Y_s^n) dA_s - \mu \int_t^T e^{\mu A_s} |Y_s^n|^2 dA_s \\ & + \int_t^T e^{\mu A_s} \|g(s)\|^2 ds + 2 \int_t^T e^{\mu A_s} S_s dK_s^n + 2 \int_t^T e^{\mu A_s} \langle Y_s^n, g(s, Y_s^n, Z_s^n) \overleftarrow{dB}_s \rangle \\ & - 2 \int_t^T e^{\mu A_s} \langle Y_s^n, Z_s^n dW_s \rangle, \end{aligned} \quad (2.7)$$

where we have used  $\int_t^T e^{\mu A_s} (Y_s^n - S_s) dK_s^n \leq 0$  and the fact that

$$\int_t^T e^{\mu A_s} Y_s^n dK_s^n = \int_t^T e^{\mu A_s} (Y_s^n - S_s) dK_s^n + \int_t^T e^{\mu A_s} S_s dK_s^n \leq \int_t^T e^{\mu A_s} S_s dK_s^n.$$

Using **(H2)** and the elementary inequality  $2ab \leq \gamma a^2 + \frac{1}{\gamma} b^2$ ,  $\forall \gamma > 0$ ,

$$\begin{aligned} 2Y_s^n f(s, Y_s^n, Z_s^n) & \leq (c\gamma_1 + \frac{1}{\gamma_1}) |Y_s^n|^2 + 2c\gamma_1 \|Z_s^n\|^2 + 2\gamma_1 f_s^2, \\ 2Y_s^n \phi(s, Y_s^n) & \leq (\gamma_2 - 2|\beta|) |Y_s^n|^2 + \frac{1}{\gamma_2} \phi_s^2. \end{aligned}$$

Taking expectation in both sides of the inequality (2.7) and choosing  $\gamma_1 = \frac{1-\alpha}{6c}$  and  $\gamma_2 - \mu = |\beta|$  we obtain for all  $\varepsilon > 0$

$$\begin{aligned} & \mathbb{E}(e^{\mu A_t} |Y_t^n|^2) + |\beta| \mathbb{E} \int_t^T e^{\mu A_s} |Y_s^n|^2 dA_s + \frac{1-\alpha}{6} \mathbb{E} \int_t^T e^{\mu A_s} \|Z_s^n\|^2 ds \\ & \leq C \mathbb{E} \left\{ e^{\mu A_T} |\xi|^2 + \int_t^T e^{\mu A_s} |Y_s^n|^2 ds + \int_t^T e^{\mu A_s} f_s^2 ds + \int_t^T e^{\mu A_s} \phi_s^2 dA_s + \int_t^T e^{\mu A_s} \|g(s)\|^2 ds \right\} \\ & + \frac{1}{\varepsilon} \mathbb{E} \left( \sup_{0 \leq s \leq T} (e^{\mu A_s} S_s^+)^2 \right) + \varepsilon \mathbb{E} (K_T^n - K_t^n)^2. \end{aligned} \quad (2.8)$$

On the other hand, we get from (2.5) that for all  $0 \leq t \leq T$ ,

$$K_T^n - K_t^n = Y_t^n - \xi - \int_t^T f(s, Y_s^n, Z_s^n) ds - \int_t^T \phi(s, Y_s^n) dA_s - \int_t^T g(s) \overleftarrow{dB}_s + \int_t^T Z_s^n dW_s. \quad (2.9)$$

Then we have

$$\begin{aligned} \mathbb{E}(K_T^n - K_t^n)^2 & \leq 6 \mathbb{E} \left\{ e^{\mu A_T} |\xi|^2 + e^{\mu A_t} |Y_t^n|^2 + \left| \int_t^T f(s, Y_s^n, Z_s^n) ds \right|^2 \right. \\ & \left. + \left| \int_t^T \phi(s, Y_s^n) dA_s \right|^2 + \left| \int_t^T g(s) \overleftarrow{dB}_s \right|^2 + \left| \int_t^T Z_s^n dW_s \right|^2 \right\}. \end{aligned} \quad (2.10)$$

It follows by Hölder inequality and the isometry equality, together with assumptions **(H2)**(a) that

$$\left| \int_t^T f(s, Y_s^n, Z_s^n) ds \right|^2 \leq 3(T-t) \int_t^T e^{\mu A_s} (f_s^2 + K^2 |Y_s^n|^2 + K^2 \|Z_s^n\|^2) ds,$$

and

$$\mathbb{E} \left| \int_t^T Z_s^n dW_s \right|^2 \leq \mathbb{E} \int_t^T e^{\mu A_s} |Z_s^n|^2 ds.$$

Next, to estimate  $\left| \int_t^T \phi(s, Y_s^n) dA_s \right|^2$ , let us assume first that  $A_T$  is a bounded real variable. For any  $\mu > 0$  given in assumptions **(H1)** or **(H2)**(a), we have

$$\begin{aligned} \left| \int_t^T \phi(s, Y_s^n) dA_s \right|^2 & \leq \left( \int_t^T e^{-\mu A_s} dA_s \right) \left( \int_t^T e^{\mu A_s} |\phi(s, Y_s^n)|^2 dA_s \right) \\ & \leq \frac{2}{\mu} \int_t^T e^{\mu A_s} (\phi_s^2 + K^2 |Y_s^n|^2) dA_s, \end{aligned}$$

since

$$\left( \int_t^T e^{-\mu A_s} dA_s \right) \leq \frac{1}{\mu} [1 - e^{-\mu A_T}] \leq \frac{1}{\mu}.$$

The general case then follows from Fatou's lemma.



Therefore, from (2.10) together with the previous inequalities, there exists a constant independent of  $A_T$  such that

$$\begin{aligned} \mathbb{E}(K_T^n - K_t^n)^2 &\leq C\mathbb{E} \left\{ e^{\mu A_T} |\xi|^2 + e^{\mu A_t} |Y_t^n|^2 + \int_t^T e^{\mu A_s} f_s^2 ds + \int_t^T e^{\mu A_s} \phi_s^2 dA_s + \int_t^T e^{\mu A_s} \|g(s)\|^2 ds \right. \\ &\quad \left. + \int_t^T e^{\mu A_s} |Y_s^n|^2 ds + \mathbb{E} \left( \sup_{0 \leq s \leq t} e^{\mu A_s} (S_s^+)^2 \right) + \int_t^T e^{\mu A_s} |Y_s^n|^2 dA_s + \int_t^T e^{\mu A_s} \|Z_s^n\|^2 ds \right\}. \end{aligned} \quad (2.11)$$

Recalling again (2.8) and taking  $\varepsilon$  small enough such that  $\varepsilon C < \min\{1, |\beta|, \frac{1-\alpha}{6}\}$ , we obtain

$$\begin{aligned} &\mathbb{E} \left( e^{\mu A_t} |Y_t^n|^2 + \int_t^T e^{\mu A_s} |Y_s^n|^2 dA_s + \int_t^T e^{\mu A_s} \|Z_s^n\|^2 ds \right) \\ &\leq C\mathbb{E} \left\{ e^{\mu A_T} |\xi|^2 + \int_t^T e^{\mu A_s} |Y_s^n|^2 ds + \int_t^T e^{\mu A_s} f_s^2 ds + \int_t^T e^{\mu A_s} \phi_s^2 dA_s \right. \\ &\quad \left. + \int_t^T e^{\mu A_s} \|g(s)\|^2 ds + \left( \sup_{0 \leq s \leq T} e^{\mu A_s} (S_s^+)^2 \right) \right\} \end{aligned}$$

Consequently, it follows from Gronwall's lemma and (2.11) that

$$\begin{aligned} &\mathbb{E} \left\{ e^{\mu A_t} |Y_t^n|^2 + \int_t^T e^{\mu A_s} |Y_s^n|^2 dA_s + \int_t^T e^{\mu A_s} \|Z_s^n\|^2 ds + |K_T^n - K_t^n|^2 \right\} \\ &\leq C\mathbb{E} \left\{ e^{\mu A_T} |\xi|^2 + \int_t^T e^{\mu A_s} f_s^2 ds + \int_t^T e^{\mu A_s} \phi_s^2 dA_s + \int_t^T e^{\mu A_s} \|g(s)\|^2 ds + \sup_{0 \leq t \leq T} e^{\mu A_t} (S_t^+)^2 \right\}. \end{aligned}$$

Finally, by application of Burkholder-Davis-Gundy inequality we obtain from (2.7)

$$\begin{aligned} \mathbb{E} \left\{ \sup_{0 \leq t \leq T} e^{\mu A_t} |Y_t^n|^2 + \int_t^T e^{\mu A_s} \|Z_s^n\|^2 ds + |K_T^n|^2 \right\} &\leq C\mathbb{E} \left\{ e^{\mu A_T} |\xi|^2 + \int_t^T e^{\mu A_s} f_s^2 ds + \int_t^T e^{\mu A_s} \phi_s^2 dA_s \right. \\ &\quad \left. + \int_t^T e^{\mu A_s} \|g(s)\|^2 ds + \sup_{0 \leq t \leq T} e^{\mu A_t} (S_t^+)^2 \right\}, \end{aligned}$$

which end the step.

*Step 2:* For each  $n \in \mathbb{N}^*$ ,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |(Y_t^n - S_t)^-|^2 \right) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Indeed, since  $Y_t^n \geq Y_t^0$ , we can w.l.o.g. replace  $S_t$  by  $S_t \vee Y_t^0$ , i.e. we may assume that  $\mathbb{E}(\sup_{0 \leq t \leq T} e^{\mu A_t} S_t^2) < \infty$ . We want to compare a.s.  $Y_t$  and  $S_t$  for all  $t \in [0, T]$ , while we do not know yet if  $Y$  is a.s. continuous.

In fact, denoting

$$\begin{cases} \bar{\xi} := \xi + \int_0^T g(s) \overleftarrow{dB}_s \\ \bar{S}_t := S_t + \int_0^t g(s) \overleftarrow{dB}_s \\ \bar{Y}_t^n := Y_t^n + \int_0^t g(s) \overleftarrow{dB}_s \end{cases}$$

we have

$$\bar{Y}_t^n = \bar{\xi} + \int_t^T f(s, Y_s^n, Z_s^n) ds + n \int_t^T (\bar{Y}_s^n - \bar{S}_s)^- ds + \int_t^T \phi(s, Y_s^n) dA_s - \int_t^T Z_s^n dW_s. \quad (2.12)$$

Set  $\sup_n \bar{Y}_t^n = \bar{Y}_t$ ; then  $Y_t = \bar{Y}_t - \int_0^t g(s) dB_s$ .

Let

$$\tilde{Y}_t^n = \bar{S}_T + \int_t^T f(s, Y_s^n, Z_s^n) ds + n \int_t^T (\bar{S}_s - \tilde{Y}_s^n) ds + \int_t^T \phi(s, Y_s^n) dA_s - \int_t^T \tilde{Z}_s^n dW_s.$$

Since  $\bar{S}_T \leq \bar{\xi}$ , then Theorem 2.1 show that, for  $t \in [0, T]$ ,  $\tilde{Y}_t^n \leq \bar{Y}_t^n$  a.s.

Let  $\sigma$  be a  $\mathcal{G}_t$ -stopping time, and put  $v = \sigma \wedge T$ . The sequence  $(\tilde{Y}_n)$  satisfies then the equality

$$\begin{aligned} \tilde{Y}_v^n &= \mathbb{E} \left\{ e^{-n(T-v)} \bar{S}_T + \int_v^T e^{-n(s-v)} f(s, Y_s^n, Z_s^n) ds + n \int_v^T e^{-n(s-v)} \bar{S}_s ds \right. \\ &\quad \left. + \int_v^T e^{-n(s-v)} \phi(s, Y_s^n) dA_s \mid \mathcal{G}_v \right\}. \end{aligned} \quad (2.13)$$

First, according to previous work (see [7]),  $e^{-n(T-v)} \bar{S}_T + n \int_v^T e^{-n(s-v)} \bar{S}_s ds$  converge to  $\bar{S}_v$  a.s. and in  $L^2(\bar{\Omega})$  and its conditional expectation converges also in  $L^2(\bar{\Omega})$ .

Moreover,

$$\begin{aligned} \mathbb{E} \left( \int_v^T e^{-n(s-v)} f(s, Y_s^n, Z_s^n) ds \right)^2 &\leq \frac{1}{2n} \mathbb{E} \left( \int_0^T |f(s, Y_s^n, Z_s^n)|^2 ds \right) \\ &\leq \frac{C}{2n} \mathbb{E} \left( \int_0^T e^{\mu A_s} (f_s^2 + |Y_s^n|^2 + \|Z_s^n\|^2) ds \right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left( \int_v^T e^{-n(s-v)} \phi(s, Y_s^n) dA_s \right)^2 &\leq \mathbb{E} \left[ \left( \int_v^T e^{-[2n(s-v) + \mu A_s]} dA_s \right) \left( \int_v^T e^{\mu A_s} |\phi(s, Y_s^n)|^2 dA_s \right) \right] \\ &\leq \frac{1}{\mu} (1 - e^{-\mu A_T}) \mathbb{E} \left( \int_0^T e^{\mu A_s} (\phi_s^2 + K |Y_s^n|^2) dA_s \right). \end{aligned}$$

Hence applying Lebesgue dominated Theorem

$$\mathbb{E} \left[ \int_v^T e^{-n(s-v)} f(s, Y_s^n, Z_s^n) ds + \int_v^T e^{-n(s-v)} \phi(s, Y_s^n) dA_s \mid \mathcal{G}_v \right] \rightarrow 0$$

in  $L^{2-\delta}(\bar{\Omega})$ , for  $\delta > 0$  arbitrary taken as  $n \rightarrow \infty$ , which implies that this convergence follows in  $L^1(\bar{\Omega})$ .

Consequently,

$$\tilde{Y}_v^n \rightarrow \bar{S}_v \quad \text{in } L^1(\bar{\Omega}) \quad \text{as } n \rightarrow \infty.$$

Therefore,  $Y_v \geq S_v$  a.s. From this and the section theorem (see [6], p.220) we deduce that  $Y_t \geq S_t$  for every  $t \in [0, T]$  and then  $(Y_t^n - S_t)^-$  converge to zero, a.s., for  $t \in [0, T]$ . Since  $(Y_t^n - S_t)^- \leq (S_t - Y_t^0)^+ \leq |S_t| + |Y_t^0|$ , the dominated convergence theorem ensures that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{0 \leq t \leq T} |(Y_t^n - S_t)^-|^2 \right) = 0.$$

*Step 3: Convergence result*

Recalling that  $Y_t^n \nearrow Y_t$  a.s, Fatou's lemma and Step 1 ensure

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\mu A_t} |Y_t|^2 \right) < +\infty.$$

It then follows by dominated convergence that

$$\mathbb{E} \left( \int_0^T |Y_s^n - Y_s|^2 ds \right) \longrightarrow 0, \text{ as } n \rightarrow \infty. \quad (2.14)$$

Next, the sequence of processes  $Z^n$  converges in  $M^2(0, T; \mathbb{R}^d)$ . Indeed, for  $n \geq p \geq 1$ , Itô's formula provide

$$\begin{aligned} & |Y_t^n - Y_t^p|^2 + \int_t^T \|Z_s^n - Z_s^p\|^2 ds \\ = & 2 \int_t^T (Y_s^n - Y_s^p) [f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)] ds + 2 \int_t^T (Y_s^n - Y_s^p) [\phi(s, Y_s^n) - \phi(s, Y_s^p)] dA_s \\ & - 2 \int_t^T \langle Y_s^n - Y_s^p, [Z_s^n - Z_s^p] dW_s \rangle + 2 \int_t^T (Y_s^n - Y_s^p) (dK_s^n - dK_s^p). \end{aligned}$$

From the same step as before, by using again assumptions **(H2)**, there exists a constant  $C > 0$ , such that

$$\begin{aligned} & \mathbb{E} \left\{ |Y_t^n - Y_t^p|^2 + \int_t^T |Y_s^n - Y_s^p|^2 dA_s + \int_t^T \|Z_s^n - Z_s^p\|^2 ds \right\} \\ \leq & C \mathbb{E} \left\{ \int_t^T |Y_s^n - Y_s^p|^2 ds + \sup_{0 \leq s \leq T} (Y_s^n - S_s)^- K_T^p + \sup_{0 \leq s \leq T} (Y_s^p - S_s)^- K_T^n \right\}, \end{aligned}$$

which, by Gronwall lemma, Hölder inequality and Step 1 implies

$$\begin{aligned} & \mathbb{E} \left\{ |Y_t^n - Y_t^p|^2 + \int_t^T |Y_s^n - Y_s^p|^2 dA_s + \int_t^T \|Z_s^n - Z_s^p\|^2 ds \right\} \\ \leq & C \left\{ \mathbb{E} \left( \sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^2 \right) \right\}^{1/2} + C \left\{ \mathbb{E} \left( \sup_{0 \leq s \leq T} |(Y_s^p - S_s)^-|^2 \right) \right\}^{1/2}. \end{aligned}$$

Finally, from Burkholder-Davis-Gundy's inequality, we obtain

$$\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y_s^n - Y_s^p|^2 + \int_t^T |Y_s^n - Y_s^p|^2 dA_s + \int_t^T \|Z_s^n - Z_s^p\|^2 ds \right) \longrightarrow 0, \text{ as } n, p \longrightarrow \infty,$$

which provides that the sequence of processes  $(Y^n, Z^n)$  is Cauchy in the Banach space  $S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$ . Consequently, there exists a couple  $(Y, Z) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$  such that

$$\mathbb{E} \left\{ \sup_{0 \leq s \leq T} |Y_s^n - Y_s|^2 + \int_t^T |Y_s^n - Y_s|^2 dA_s + \int_t^T \|Z_s^n - Z_s\|^2 ds \right\} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

On the other hand, we rewrite (2.9) as

$$K_t^n = Y_0^n - Y_t^n - \int_0^t f(s, Y_s^n, Z_s^n) ds - \int_0^t \phi(s, Y_s^n) dA_s - \int_0^t g(s) \overleftarrow{dB}_s + \int_0^t Z_s^n dW_s. \quad (2.15)$$

By the convergence of  $Y^n, Z^n$  (for a subsequence), the fact that  $f, \phi$  are continuous and

- $\sup_{n \geq 0} |f(s, Y_s^n, Z_s)| \leq f_s + K \{(\sup_{n \geq 0} |Y_s^n|) + \|Z_s\|\},$
- $\sup_{n \geq 0} |\phi(s, Y_s^n)| \leq \phi_s + K \{(\sup_{n \geq 0} |Y_s^n|)\},$
- $\mathbb{E} \int_t^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s)|^2 ds \leq C \mathbb{E} \int_t^T \|Z_s^n - Z_s\|^2 ds$

we get the existence of a process  $K$  which verifies for all  $t \in [0, T]$

$$\mathbb{E} |K_t^n - K_t|^2 \longrightarrow 0$$

and passing to the limit in (2.5), we have

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T \phi(s, Y_s) dA_s + K_T - K_t + \int_t^T g(s) \overleftarrow{dB}_s - \int_t^T Z_s dW_s, 0 \leq t \leq T,$$

$\mathbb{P}$ -a.s.

It remains to show that  $(Y, Z, K)$  solves the reflected BSDE  $(\xi, f, \phi, g, S)$ . In fact, since  $(Y^n, K^n)$  converges to  $(Y, K)$  in probability uniformly in  $t$ ,  $dK^n$  converges to  $dK$  in probability. This implies  $\int_0^T (Y_s^n - S_s) dK_s^n$  converges to  $\int_0^T (Y_s - S_s) dK_s$  in probability. We have  $Y_t \geq S_t$  a.s. for  $t \in [0, T]$  so that  $\int_0^T (Y_s - S_s) dK_s \geq 0$ . On the other hand,  $\int_0^T (Y_s^n - S_s) dK_s^n = -n \int_0^T |(Y_s^n - S_s)^-|^2 ds \leq 0$ . Hence  $\int_0^T (Y_s - S_s) dK_s = 0$ .

### Uniqueness

Let us define

$$\{(\Delta Y_t, \Delta Z_t, \Delta K_t), 0 \leq t \leq T\} = \{(Y_t - Y'_t, Z_t - Z'_t, K_t - K'_t), 0 \leq t \leq T\}$$

where  $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$  and  $\{(Y'_t, Z'_t, K'_t), 0 \leq t \leq T\}$  denote two solutions of the reflected generalized BDSDE (2.3).

It follows again by Itô's formula that for every  $0 \leq t \leq T$

$$\begin{aligned} |\Delta Y_t|^2 + \int_t^T \|\Delta Z_s\|^2 ds &= 2 \int_t^T \Delta Y_s (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) ds \\ &\quad + 2 \int_t^T \Delta Y_s (\phi(s, Y_s) - \phi(s, Y'_s)) dA_s \\ &\quad - 2 \int_t^T \langle \Delta Y_s, \Delta Z_s dW_s \rangle + 2 \int_t^T \Delta Y_s d(\Delta K_s). \end{aligned}$$

Since

$$\int_t^T \Delta Y_s d(\Delta K_s) \leq 0,$$

and by using similar computation as in the proof of existence, we have

$$\mathbb{E} \left\{ |\Delta Y_t|^2 + \int_t^T |\Delta Y_s| dA_s + \int_t^T \|\Delta Z_s\|^2 ds \right\} \leq C \mathbb{E} \int_t^T |\Delta Y_s|^2 ds,$$

from which, we deduce that  $\Delta Y_t = 0$  and further  $\Delta Z_t = 0$ . Moreover since

$$\begin{aligned} \Delta K_t &= \Delta Y_0 - \Delta Y_t - \int_0^t (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) ds - \int_0^t (\phi(s, Y_s) - \phi(s, Y'_s)) dA_s \\ &\quad + \int_0^t \Delta Z_s dW_s, \end{aligned}$$

we have  $\Delta K_t = 0$ . The proof is now complete.  $\square$

*Proof of Theorem 2.2.* For any given  $(\bar{Y}, \bar{Z}) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$ , let consider the reflected generalized BDSDE:

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s) ds + \int_t^T \phi(s, Y_s) dA_s + \int_t^T g(s, \bar{Y}_s, \bar{Z}_s) \overleftarrow{dB}_s - \int_t^T Z_s dW_s + K_T - K_t. \quad (2.16)$$

It follows from Proposition 2.1 that reflected generalized BDSDE (2.16) has a unique solution  $(Y, Z, K)$ . Therefore, the mapping

$$\begin{aligned} \Phi : S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d) &\longrightarrow S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d) \\ (\bar{Y}, \bar{Z}) &\longmapsto (Y, Z) = \Phi(\bar{Y}, \bar{Z}). \end{aligned}$$

is well defined.

Next, let  $(Y, Z), (Y', Z'), (\bar{Y}, \bar{Z})$  and  $(\bar{Y}', \bar{Z}') \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$  such that  $(Y, Z) = \Phi(\bar{Y}, \bar{Z})$  and  $(Y', Z') = \Phi(\bar{Y}', \bar{Z}')$  and set  $\Delta \eta = \eta - \eta'$  for  $\eta = Y, \bar{Y}, Z, \bar{Z}$ . By virtue of Itô's formula, we have

$$\begin{aligned} &\mathbb{E} e^{\mu t + \beta A_t} |\Delta Y_t|^2 + \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|\Delta Z_s\|^2 ds \\ &= 2\mathbb{E} \int_t^T e^{\mu s + \beta A_s} \Delta Y_s \{f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)\} ds + 2\mathbb{E} \int_t^T e^{\mu s + \beta A_s} \Delta Y_s \{\phi(s, Y_s) - \phi(s, Y'_s)\} dA_s \\ &\quad + 2\mathbb{E} \int_t^T e^{\mu s + \beta A_s} \Delta Y_s d(\Delta K_s) + \int_t^T e^{\mu s + \beta A_s} \|g(s, \bar{Y}_s, \bar{Z}_s) - g(s, \bar{Y}'_s, \bar{Z}'_s)\|^2 ds \\ &\quad - \mu \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 ds - \beta \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 dA_s. \end{aligned}$$

From **(H2)** there exists  $\alpha < \alpha' < 1$  such that

$$\begin{aligned} \Delta Y_s \{f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)\} &\leq \left[ \frac{c}{1 - \alpha'} + (1 - \alpha') \right] |\Delta Y_s|^2 + (1 - \alpha') \|\Delta Z_s\|^2 \\ \Delta Y_s \{\phi(s, Y_s) - \phi(s, Y'_s)\} &\leq \beta |\Delta Y_s|^2, \end{aligned}$$

which together with  $\mathbb{E} \int_t^T e^{\mu s + \beta A_s} \Delta Y_s d(K_s - K'_s) \leq 0$ , provide

$$\begin{aligned} & \mathbb{E} e^{\mu t + \beta A_t} |\Delta Y_t|^2 + \alpha' \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|\Delta Z_s\|^2 ds \\ & \leq \left( \frac{c}{1 - \alpha'} + 1 - \alpha' - \mu \right) \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 ds + \beta \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 dA_s \\ & \quad + c \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta \bar{Y}_s|^2 ds + \alpha \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|\Delta \bar{Z}_s\|^2 ds \end{aligned}$$

Next, choosing  $\mu$  such that  $\mu - \frac{c}{1 - \alpha'} - 1 + \alpha' = \frac{\alpha' c}{\alpha}$ , we obtain

$$\begin{aligned} & \alpha' \left[ \frac{c}{\alpha} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 ds + \frac{|\beta|}{\alpha'} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 dA_s + \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|\Delta Z_s\|^2 ds \right] \\ & \leq \alpha \left( \frac{c}{\alpha} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta \bar{Y}_s|^2 ds + \frac{|\beta|}{\alpha'} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta \bar{Y}_s|^2 dA_s + \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|\Delta \bar{Z}_s\|^2 ds \right). \end{aligned}$$

Therefore

$$\begin{aligned} & \bar{c} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 ds + \bar{\beta} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta Y_s|^2 dA_s + \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|\Delta Z_s\|^2 ds \\ & \leq \frac{\alpha}{\alpha'} \left( \bar{c} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta \bar{Y}_s|^2 ds + \bar{\beta} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |\Delta \bar{Y}_s|^2 dA_s + \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|\Delta \bar{Z}_s\|^2 ds \right) \end{aligned}$$

where  $\bar{c} = c/\alpha$  and  $\bar{\beta} = |\beta|/\alpha'$ .

Since  $\frac{\alpha}{\alpha'} < 1$ , then  $\Phi$  is a strict contraction on  $\mathcal{S}^2([0, T], \mathbb{R}) \times \mathcal{M}^2((0, T); \mathbb{R}^d)$  equipped with the norm

$$\|(Y, Z)\|^2 = \bar{c} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |Y_s|^2 ds + \bar{\beta} \mathbb{E} \int_t^T e^{\mu s + \beta A_s} |Y_s|^2 dA_s + \mathbb{E} \int_t^T e^{\mu s + \beta A_s} \|Z_s\|^2 ds.$$

Then it has a unique fixed point, which is the unique solution of BDSDE (2.1).  $\square$

### 3 Connection to stochastic viscosity solution for reflected SPDEs with nonlinear Neumann boundary condition

In this section we will investigate the reflected generalized BDSDEs studied in the previous section in order to give a interpretation for the stochastic viscosity solution of a class of nonlinear reflected SPDEs with nonlinear Neumann boundary condition.

#### 3.1 Notion of stochastic viscosity solution for reflected SPDEs with nonlinear Neumann boundary condition

With the same notations as in Section 2, let  $\mathbf{F}^B = \{\mathcal{F}_{t,T}^B\}_{0 \leq t \leq T}$  be the filtration generated by  $B$ . The set  $\mathcal{M}_{0,T}^B$  denote all the  $\mathbf{F}^B$ -stopping times  $\tau$  such  $0 \leq \tau \leq T$ , a.s. For generic Euclidean spaces  $E$  and  $E_1$  we introduce the following:

1. The symbol  $C^{k,n}([0, T] \times E; E_1)$  stands for the space of all  $E_1$ -valued functions defined on  $[0, T] \times E$  which are  $k$ -times continuously differentiable in  $t$  and  $n$ -times continuously differentiable in  $x$ , and  $C_b^{k,n}([0, T] \times E; E_1)$  denotes the subspace of  $C^{k,n}([0, T] \times E; E_1)$  in which all functions have uniformly bounded partial derivatives.
2. For any sub- $\sigma$ -field  $\mathcal{G} \subseteq \mathcal{F}_T^B$ ,  $C^{k,n}(\mathcal{G}, [0, T] \times E; E_1)$  (resp.  $C_b^{k,n}(\mathcal{G}, [0, T] \times E; E_1)$ ) denotes the space of all  $C^{k,n}([0, T] \times E; E_1)$  (resp.  $C_b^{k,n}([0, T] \times E; E_1)$ )-valued random variable that are  $\mathcal{G} \otimes \mathcal{B}([0, T] \times E)$ -measurable;
3.  $C^{k,n}(\mathbf{F}^B, [0, T] \times E; E_1)$  (resp.  $C_b^{k,n}(\mathbf{F}^B, [0, T] \times E; E_1)$ ) is the space of all random fields  $\phi \in C^{k,n}(\mathcal{F}_T, [0, T] \times E; E_1)$  (resp.  $C_b^{k,n}(\mathcal{F}_T, [0, T] \times E; E_1)$ ), such that for fixed  $x \in E$  and  $t \in [0, T]$ , the mapping  $\omega \mapsto \alpha(t, \omega, x)$  is  $\mathbf{F}^B$ -progressively measurable.
4. For any sub- $\sigma$ -field  $\mathcal{G} \subseteq \mathcal{F}^B$  and a real number  $p \geq 0$ ,  $L^p(\mathcal{G}; E)$  to be all  $E$ -valued  $\mathcal{G}$ -measurable random variable  $\xi$  such that  $\mathbb{E}|\xi|^p < \infty$ .

Furthermore, regardless their dimensions we denote by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  the inner product and norm in  $E$  and  $E_1$ , respectively. For  $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ , we denote  $D_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ ,  $D_{xx} = (\partial_{x_i x_j}^2)_{i,j=1}^d$ ,  $D_y = \frac{\partial}{\partial y}$ ,  $D_t = \frac{\partial}{\partial t}$ . The meaning of  $D_{xy}$  and  $D_{yy}$  is then self-explanatory.

Let  $\Theta$  be an open connected and smooth bounded domain of  $\mathbb{R}^d$  ( $d \geq 1$ ) such that for a function  $\psi \in C_b^2(\mathbb{R}^d)$ ,  $\Theta$  and its boundary  $\partial\Theta$  are characterized by  $\Theta = \{\psi > 0\}$ ,  $\partial\Theta = \{\psi = 0\}$  and, for any  $x \in \partial\Theta$ ,  $\nabla\psi(x)$  is the unit normal vector pointing towards the interior of  $\Theta$ .

In this section, we shall make use of the following standing assumptions:

- (A1) The functions  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  and  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  are uniformly Lipschitz continuous, with a common Lipschitz constant  $K > 0$ .
- (A2) The functions  $f : \Omega \times [0, T] \times \overline{\Theta} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\phi : \Omega \times [0, T] \times \overline{\Theta} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous random field such that for fixed  $(x, y, z)$  and  $f(\cdot, x, y, z)$ ,  $\phi(\cdot, x, y)$  and  $h(\cdot, x)$  are  $\mathbf{F}^B$ -progressively measurable; and there exists  $K > 0$ , such that for  $\mathbb{P}$ -a.e  $\omega$ ,
- (i)  $|f(\omega, t, x, y, z)| \leq K(1 + |x| + |y| + \|z\|)$ ,
  - (ii)  $|f(\omega, t, x, y, z) - f(\omega, t', x', y', z')| \leq c(|t - t'| + |x - x'| + |y - y'| + \|z - z'\|)$ ,
  - (iii)  $|\phi(\omega, t, x, y)| \leq K(1 + |x| + |y|)$ ,
  - (iv)  $\langle y - y', \phi(\omega, t, x, y) - \phi(\omega, t, x, y') \rangle \leq \beta |y - y'|^2$
  - (v)  $|\phi(\omega, t, x, y) - \phi(\omega, t, x, y')| \leq K(|x - x'| + |y - y'|)$
- (A3) The function  $l : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous, such that for some constants  $K, p > 0$

$$|l(x)| \leq K(1 + |x|^p), \quad x \in \mathbb{R}^n.$$

- (A4) The function  $h : \Omega \times [0, T] \times \overline{\Theta} \rightarrow \mathbb{R}$  is continuous random field such that for fixed  $x$ ,  $h(\cdot, x)$  is  $\mathbf{F}^B$ -progressively measurable; and there exists  $K > 0$ , such that for  $\mathbb{P}$ -a.e  $\omega$ ,

$$(i) |h(\omega, t, x)| \leq K(1 + |x|),$$

$$(ii) h(\omega, T, x) \leq l(x).$$

(A5) The function  $g \in C_b^{0,2,3}([0, T] \times \bar{\Theta} \times \mathbb{R}; \mathbb{R})$ .

Let us consider the related obstacle problem for SPDE with nonlinear Neumann boundary condition:

$$\mathcal{OP}^{(f, \phi, g, h, l)} \left\{ \begin{array}{l} (i) \min \left\{ u(t, x) - h(t, x), -\frac{\partial u(t, x)}{\partial t} - [Lu(t, x) + f(t, x, u(t, x), \sigma^*(x) D_x u(t, x))] \right. \\ \quad \left. - g(t, x, u(t, x)) \dot{B}_s \right\} = 0, \quad (t, x) \in [0, T] \times \Theta \\ (ii) \frac{\partial u}{\partial n}(t, x) + \phi(t, x, u(t, x)) = 0, \quad (t, x) \in [0, T] \times \partial\Theta, \\ (iii) u(T, x) = l(x), \quad x \in \bar{\Theta} \end{array} \right.$$

where

$$L = \frac{1}{2} \sum_{i,j=1}^d (\sigma(x) \sigma^*(x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}, \quad \forall x \in \Theta,$$

and

$$\frac{\partial}{\partial n} = \sum_{i=1}^d \frac{\partial \psi}{\partial x_i}(x) \frac{\partial}{\partial x_i}, \quad \forall x \in \partial\Theta.$$

*Remark 3.1.* In the previous definition, the stochastic “variational inequality” is defined formally since it involves a quantitative comparison between a random field and its stochastic differential. Therefore it does actually make sense as follows (see [10]): there exists a regular random measure  $\nu$  such that (i) becomes

$$\left\{ \begin{array}{l} (iv) u(t, x) \geq h(t, x), \quad d\mathbb{P} \otimes dt \otimes dx - a.e., \\ (v) -\frac{\partial u}{\partial t}(t, x) - [Lu(t, x) + f(t, x, u(t, x), \sigma^*(x) \nabla u(t, x))] - g(t, x, u(t, x)) \dot{B}_s = -\nu(dt, dx), \\ a.s., (t, x) \in [0, T] \times \Theta, \\ (vi) \nu(u > h) = 0, \quad a.s. \end{array} \right.$$

Our next goal is to define the notion of stochastic viscosity solution to  $\mathcal{OP}^{(f, \phi, g, h, l)}$ . In fact, we recall some of the notations appear in [2]. Let  $\eta \in \mathcal{C}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d \times \mathbb{R})$  be the solution of equation

$$\eta(t, x, y) = y + \int_t^T \langle g(s, x, \eta(s, x, y)), \circ \overleftarrow{dB}_s \rangle,$$



where  $\circ\overleftarrow{dB}$  is the Stratonowich backward stochastic integral which respect the Brownian  $B$ . We have equivalence with Itô backward stochastic integral which respect the Brownian  $B$  as follows:

$$\int_t^T \langle g(s, x, \eta(s, x, y)), \circ\overleftarrow{dB}_s \rangle = \frac{1}{2} \int_t^T \langle g, D_y g \rangle (s, x, \eta(s, x, y)) ds + \int_t^T \langle g(s, x, \eta(s, x, y)), \overleftarrow{dB}_s \rangle.$$

By (A5) and for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , the mapping  $y \mapsto \eta(s, x, y)$  defines a diffeomorphism almost surely. Hence if we denote by  $\varepsilon(s, x, y)$  its  $y$ -inverse, one can show that (cf. Buckdahn and Ma [2])

$$\varepsilon(t, x, y) = y - \int_t^T \langle D_y \varepsilon(s, x, y) g(s, x, y), \circ\overleftarrow{dB}_s \rangle. \quad (3.1)$$

To simplify the notation in the sequel we denote

$$A_{f,g}(\varphi(t, x)) = -L\varphi(t, x) - f(t, x, \varphi(t, x), \sigma^* D_x \varphi(t, x)) + \frac{1}{2} (g, D_y g)(t, x, \varphi(t, x))$$

and  $\Psi(t, x) = \eta(t, x, \varphi(t, x))$ .

**Definition 3.1.** A random field  $u \in C(\mathbf{F}^B, [0, T] \times \overline{\Theta})$  is called a stochastic viscosity sub-solution of the stochastic obstacle problem  $\mathcal{OP}^{(f, \phi, g, h, l)}$  if  $u(T, x) \leq l(x)$ , for all  $x \in \overline{\Theta}$ , and if for any stopping time  $\tau \in \mathcal{M}_{0,T}^B$ , any state variable  $\xi \in L^0(\mathcal{F}_\tau^B, \overline{\Theta})$ , and any random field  $\varphi \in C^{1,2}(\mathcal{F}_\tau^B, [0, T] \times \mathbb{R}^d)$  such that for  $\mathbb{P}$ -almost all  $\omega \in \{0 < \tau < T\}$ ,

$$u(t, \omega, x) - \Psi(t, \omega, x) \leq 0 = u(\tau(\omega), \xi(\omega)) - \Psi(\tau(\omega), \xi(\omega))$$

for all  $(t, x)$  in some neighborhood  $\mathcal{V}(\omega, \tau(\omega), \xi(\omega))$  of  $(\tau(\omega), \xi(\omega))$ , it holds:

(a) on the event  $\{0 < \tau < T\} \cap \{\xi \in \Theta\}$

$$\min \{u(\tau, \xi) - h(\tau, \xi), A_{f,g}(\Psi(\tau, \xi)) - D_y \Psi(\tau, \xi) D_t \varphi(\tau, \xi)\} \leq 0 \quad (3.2)$$

holds,  $\mathbb{P}$ -almost surely;

(b) on the event  $\{0 < \tau < T\} \cap \{\xi \in \partial\Theta\}$  the inequality

$$\min \left[ \min \{u(\tau, \xi) - h(\tau, \xi), A_{f,g}(\Psi(\tau, \xi)) - D_y \Psi(\tau, \xi) D_t \varphi(\tau, \xi)\}, \right. \\ \left. - \frac{\partial \Psi}{\partial n}(\tau, \xi) - \phi(\tau, \xi, \Psi(\tau, \xi)) \right] \leq 0 \quad (3.3)$$

holds,  $\mathbb{P}$ -almost surely.

A random field  $u \in C(\mathbf{F}^B, [0, T] \times \overline{\Theta})$  is called a stochastic viscosity supersolution of the stochastic obstacle problem  $\mathcal{OP}^{(f, \phi, g, h, l)}$  if  $u(T, x) \geq l(x)$ , for all  $x \in \overline{\Theta}$ , and if for any stopping time  $\tau \in \mathcal{M}_{0,T}^B$ , any state variable  $\xi \in L^0(\mathcal{F}_\tau^B, \overline{\Theta})$ , and any random field  $\varphi \in C^{1,2}(\mathcal{F}_\tau^B, [0, T] \times \mathbb{R}^d)$  such that for  $\mathbb{P}$ -almost all  $\omega \in \{0 < \tau < T\}$

$$u(t, \omega, x) - \Psi(t, \omega, x) \geq 0 = u(\tau(\omega), \xi(\omega)) - \Psi(\tau(\omega), \xi(\omega))$$

for all  $(t, x)$  in some neighborhood  $\mathcal{V}(\omega, \tau(\omega), \xi(\omega))$  of  $(\tau(\omega), \xi(\omega))$ , it holds:

(a) on the event  $\{0 < \tau < T\} \cap \{\xi \in \Theta\}$

$$\min \{u(\tau, \xi) - h(\tau, \xi), A_{f,g}(\Psi(\tau, \xi)) - D_y \Psi(\tau, \xi) D_t \phi(\tau, \xi)\} \geq 0 \quad (3.4)$$

holds,  $\mathbb{P}$ -almost surely;

(b) on the event  $\{0 < \tau < T\} \cap \{\xi \in \partial\Theta\}$

$$\max \left[ \min \{u(\tau, \xi) - h(\tau, \xi), A_{f,g}(\Psi(\tau, \xi)) - D_y \Psi(\tau, \xi) D_t \phi(\tau, \xi)\}, \right. \\ \left. - \frac{\partial \Psi}{\partial n}(\tau, \xi) - \phi(\tau, \xi, \Psi(\tau, \xi)) \right] \geq 0 \quad (3.5)$$

holds,  $\mathbb{P}$ -almost surely.

Finally, a random field  $u \in C(\mathbf{F}^B, [0, T] \times \overline{\Theta})$  is called a stochastic viscosity solution of the stochastic obstacle problem  $\mathcal{OP}^{(f, \phi, g, h, l)}$  if it is both a stochastic viscosity subsolution and a stochastic viscosity supersolution.

*Remark 3.2.* Observe that if  $f, \phi$  are deterministic and  $g \equiv 0$ , the flow  $\eta$  becomes  $\eta(t, x, y) = y$  and  $\Psi(t, x) = \phi(t, x)$ ,  $\forall (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ . Thus, definition 3.1 coincides with the definition of (deterministic) viscosity solution of PDE  $\mathcal{OP}^{(f, \phi, 0, h, l)}$  given in [15] for each fixed  $\omega \in \{0 < \tau < T\}$ , modulo the  $\mathbf{F}^B$ -measurability requirement of the test function  $\phi$ .

Now let us recall a notion of random viscosity solution which will be a bridge linking the stochastic viscosity solution and its deterministic counterpart.

**Definition 3.2.** A random field  $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^n)$  is called an  $\omega$ -wise viscosity solution if for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ ,  $u(\omega, \cdot, \cdot)$  is a (deterministic) viscosity solution of  $\mathcal{OP}^{(f, \phi, 0, h, l)}$ .

Next we introduce the Doss-Sussman transformation. It enables us to convert an reflected SPDE of the form  $\mathcal{OP}^{(f, \phi, g, h, l)}$  to an classical partial differential equation of the form  $\mathcal{OP}^{(\tilde{f}, 0, \tilde{\phi}, \tilde{h})}$  where  $\tilde{f}, \tilde{\phi}$  and  $\tilde{h}$  are certain well-defined random fields, which are defined in terms of  $f, \phi$  and  $h$ .

**Proposition 3.1.** Assume (A1)-(A5). A random field  $u$  is a stochastic viscosity solution to the  $\mathcal{OP}^{(f, \phi, g, h, l)}$  if and only if  $v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot))$  is a stochastic viscosity solution to the SPDE  $\mathcal{OP}^{(\tilde{f}, 0, \tilde{\phi}, \tilde{h}, l)}$ , where  $(\tilde{f}, \tilde{\phi}, \tilde{h})$  are three coefficients that will be made precise later (see (3.11), (3.12) and (3.6)).

*Proof.* We shall only prove that if  $u \in C(\mathbf{F}^B, [0, T] \times \overline{\Theta})$  is a stochastic viscosity sub-(resp. super-)solution to SPDE  $\mathcal{OP}^{(f, \phi, g, h, l)}$ , then  $v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot))$  belongs to  $C(\mathbf{F}^B, [0, T] \times \overline{\Theta})$ , and it is a stochastic viscosity sub-(resp. super-) solution to SPDE  $\mathcal{OP}^{(\tilde{f}, \tilde{\phi}, 0, \tilde{h}, l)}$ . The converse part of the proposition can be proved in a very similar way. We shall only discuss for the stochastic subsolution case, as the supersolution part can be proved similarly. Therefore, let us assume that  $u \in C(\mathbf{F}^B, [0, T] \times \overline{\Theta})$  is the is a stochastic viscosity subsolution of the SPDE  $\mathcal{OP}^{(f, \phi, g, h, l)}$ . It then follows that  $v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot))$  belongs to  $C(\mathbf{F}^B, [0, T] \times \overline{\Theta})$ . Let now show that  $v$  is a stochastic viscosity subsolution of the SPDE

$\mathcal{O}P(\tilde{f}, 0, \tilde{\phi}, \tilde{h}, l)$ . Firstly, since  $y \mapsto \varepsilon(\cdot, \cdot, y)$  is increasing,  $v(t, x) \geq \tilde{h}(t, x), \forall (t, x) \in [0, T] \times \overline{\Theta}$ , where

$$\tilde{h}(t, x) = \varepsilon(t, x, h(t, x)). \quad (3.6)$$

Following, there exist  $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^0(\mathcal{F}_\tau^B; \overline{\Theta})$  satisfy

$$\mathbb{P}(v(\tau, \xi) > \tilde{h}(\tau, \xi), 0 < \tau < T) > 0; \quad (3.7)$$

and  $\varphi \in C^{1,2}(\mathcal{F}_\tau^B, [0, T] \times \overline{\Theta})$  such that for  $\mathbb{P}$ -almost all  $\omega \in \{0 < \tau < T, v(\tau, \xi) > \tilde{h}(\tau, \xi)\}$ , the inequality

$$u(\omega, t, x) - \Psi(\omega, t, x) \leq 0 = u(\omega, \tau(\omega), \xi(\omega)) - \Psi(\omega, \tau(\omega), \xi(\omega)) \quad (3.8)$$

holds for all  $(t, x)$  in some neighborhood  $\mathcal{V}(\omega, \tau(\omega), \xi(\tau))$  of  $(\omega, \tau(\omega))$ . Next putting  $\Psi(t, x) = \eta(t, x, \varphi(t, x))$  and since the mapping  $y \mapsto \eta(t, x, y)$  is strictly increasing, for all  $(t, x) \in \mathcal{V}(\tau, \xi)$  we get that

$$\begin{aligned} u(t, x) - \Psi(t, x) &= \eta(t, x, v(t, x)) - \eta(t, x, \varphi(t, x)) \\ &\leq 0 = \eta(\tau, v(\tau, \xi)) - \eta(\tau, \varphi(\tau, \xi)) \\ &= u(\tau, \xi) - \Psi(\tau, \xi) \end{aligned}$$

holds  $\mathbb{P}$ -almost surely on  $\omega \in \{0 < \tau < T, v(\tau, \xi) > \tilde{h}(\tau, \xi)\}$ . According to (3.7) and recall again the strictly increasing of the mapping  $y \mapsto \eta(t, x, y)$ , we have  $\{v(\tau, \xi) > \tilde{h}(\tau, \xi)\} = \{u(\tau, \xi) > h(\tau, \xi)\}$ . Moreover, since  $u$  is a stochastic viscosity sub-solution of the SPDE  $\mathcal{O}P^{(f, \phi, g, h, l)}$ , we obtain:

(a) on the event  $\{0 < \tau < T\} \cap \{u(\tau, \xi) > h(\tau, \xi)\} \cap \{\xi \in \Theta\}$

$$A_{f,g}(\Psi(\tau, \xi)) - D_y \Psi(\tau, \xi) D_t \varphi(\tau, \xi) \leq 0$$

holds,  $\mathbb{P}$ -almost surely;

(b) on the event  $\{0 < \tau < T\} \cap \{u(\tau, \xi) > h(\tau, \xi)\} \cap \{\xi \in \partial\Theta\}$  the inequality

$$\min \left[ A_{f,g}(\Psi(\tau, \xi)) - D_y \Psi(\tau, \xi) D_t \varphi(\tau, \xi), -\frac{\partial \Psi}{\partial n}(\tau, \xi) - \phi(\tau, \xi, \Psi(\tau, \xi)) \right] \leq 0$$

holds,  $\mathbb{P}$ -almost surely.

By the similarly calculation used in [1], we have:

(a) on the event  $\{0 < \tau < T\} \cap \{v(\tau, \xi) > \tilde{h}(\tau, \xi)\} \cap \{\xi \in \Theta\}$  the inequality

$$A_{\tilde{f},0}(\varphi(\tau, \xi)) - D_t \varphi(\tau, \xi) \leq 0 \quad (3.9)$$

holds,  $\mathbb{P}$ -almost surely;

(b) on the event  $\{0 < \tau < T\} \cap \{v(\tau, \xi) > \tilde{h}(\tau, \xi)\} \cap \{\xi \in \partial\Theta\}$  the inequality

$$\min \left[ A_{\tilde{f},0}(\varphi(\tau, \xi)) - D_t \varphi(\tau, \xi), -\frac{\partial \varphi}{\partial n}(\tau, \xi) - \tilde{\phi}(\tau, \xi, \varphi(\tau, \xi)) \right] \leq 0 \quad (3.10)$$

holds,  $\mathbb{P}$ -almost surely.

where

$$\begin{aligned} \tilde{f}(t, x, y, z) = & \frac{1}{D_y \eta(t, x, y)} \left[ f(t, x, \eta(t, x, y), \sigma(x)^* D_x \eta(t, x, y) + D_y \eta(t, x, y) z) \right. \\ & - \frac{1}{2} g D_y g(t, x, \eta(t, x, y)) + L_x \eta(t, x, y) + \langle \sigma(x)^* D_{xy} \eta(t, x, y), z \rangle \\ & \left. + \frac{1}{2} D_{yy} \eta(t, x, y) |z|^2 \right] \end{aligned} \quad (3.11)$$

and

$$\tilde{\phi}(t, x, y) = \frac{1}{D_y \eta(t, x, y)} [h(t, x, \eta(t, x, y)) + D_x \eta(t, x, y) \nabla \psi(x)]. \quad (3.12)$$

Combining inequality (3.9) and (3.10), we obtain that the random field  $v$  is a stochastic viscosity subsolution of the SPDE  $\mathcal{O}\mathcal{P}^{(\tilde{f}, \tilde{\phi}, 0, \tilde{h}, l)}$ , which ends the proof of Proposition 3.1.  $\square$

### 3.2 Existence of stochastic viscosity solutions for SPDE with nonlinear Neumann boundary condition

The main objective of this subsection is to give a link between the stochastic obstacle problem  $\mathcal{O}\mathcal{P}^{(f, \phi, g, h, l)}$  and the reflected generalized BDSDE (2.1) introduced in Section 1. We consider

$$\begin{aligned} s \mapsto A_s^{t,x} \text{ is increasing} \\ X_s^{t,x} = x + \int_t^{s \vee t} b(X_r^{t,x}) dr + \int_t^{s \vee t} \sigma(X_r^{t,x}) dW_r + \int_t^{s \vee t} \nabla \psi(X_r^{t,x}) dA_r^{t,x}, \quad \forall s \in [t, T], \\ A_s^{t,x} = \int_t^{s \vee t} I_{\{X_r^{t,x} \in \partial\Theta\}} dA_r^{t,x}. \end{aligned} \quad (3.13)$$

It is clear (see [9]) that under conditions **(A1)** on the coefficients  $b$  and  $\sigma$ , (3.13) has a unique strong  $\mathbf{F}^W$ -adapted solution.

Using the similar arguments as in Pardoux and Zhang [14] (Propositions 3.1 and 3.2), or Slomiński [16], we can provide the following regularity results.

**Proposition 3.2.** *There exists a constant  $C > 0$  such that for all for all  $0 \leq t < t' \leq T$  and  $x, x' \in \overline{\Theta}$ , the following inequalities hold: for any  $p > 4$*

$$\mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X_s^{t,x} - X_s^{t',x'} \right|^p \right] \leq C \left\{ |t' - t|^{p/2} + |x - x'|^p \right\}$$

and

$$\mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| A_s^{t,x} - A_s^{t',x'} \right|^p \right] \leq C \left\{ |t' - t|^{p/2} + |x - x'|^p \right\}.$$

Moreover, for all  $p \geq 1$ , there exists a constant  $C_p$  such that for all  $(t, x) \in [0, T] \times \overline{\Theta}$ ,

$$\mathbb{E}(|A_s^{t,x}|^p) \leq C_p(1+t^p)$$

and for each  $\mu$ ,  $t < s < T$ , there exists a constant  $C(\mu, t)$  such that for all  $x \in \overline{\Theta}$ ,

$$\mathbb{E}(e^{\mu A_s^{t,x}}) \leq C(\mu, t).$$

We consider also the following reflected generalized BDSDE: for  $(t, x) \in [0, T] \times \overline{\Theta}$

$$\begin{cases} Y_s^{t,x} = l(X_T^{t,x}) + \int_{s \vee t}^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_{s \vee t}^T g(r, X_r^{t,x}, Y_r^{t,x}) \overleftarrow{d}B_r \\ \quad + \int_{s \vee t}^T \phi(r, X_r^{t,x}, Y_r^{t,x}) dA_r^{t,x} + K_T^{t,x} - K_{s \vee t}^{t,x} - \int_{s \vee t}^T Z_r^{t,x} dW_r, \\ Y_s^{t,x} \geq h(s, X_s^{t,x}) \text{ such that } \int_{s \vee t}^T (Y_r^{t,x} - h(r, X_r^{t,x})) dK_r^{t,x} = 0. \end{cases} \quad (3.14)$$

where the coefficients  $l$ ,  $f$ ,  $g$ ,  $\phi$  and  $h$  satisfy the hypotheses (A2)-(A5). The following regularity result generalizes the Kolmogorov continuity criterion to BDSDEs:

**Proposition 3.3.** *Let the ordered triplet  $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$  be a solution of the BDSDE (3.14). Then the random field  $(s, t, x) \mapsto Y_s^{t,x}$  is almost surely continuous on  $[0, T] \times [0, T] \times \overline{\Theta}$ .*

*Proof.* In the same way as in the proof of Lemma 2.1, we have, for  $t, t' \in [0, T]$ ,  $x, x' \in \Theta$  and  $p > 4$ ,

$$\begin{aligned} & \mathbb{E} \left( \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p \right) + \mathbb{E} |K_T^{t,x} - K_T^{t',x'}|^p + \mathbb{E} \left( \int_0^T |Z_r^{t,x} - Z_r^{t',x'}|^2 dr \right)^{p/2} + \mathbb{E} \left( \int_0^T |Y_r^{t,x} - Y_r^{t',x'}|^p dA_r^{t,x} \right) \\ & \leq C \left[ \mathbb{E} \left( \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p \right) + \mathbb{E} \left( \int_0^T [\mathbf{1}_{[t,T]} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - \mathbf{1}_{[t',T]} f(r, X_r^{t',x'}, Y_r^{t',x'}, Z_r^{t',x'})] dr \right) \right. \\ & \quad + \mathbb{E} \left( \int_{t \wedge t'}^{t \vee t'} |\phi(r, X_r^{t,x}, Y_r^{t,x})|^p dA_r^{t,x} \right) + \mathbb{E} \left( \int_{t \wedge t'}^{t \vee t'} |h(r, X_r^{t,x})| [dK_r^{t,x} + dK_r^{t',x'}] \right) + \mathbb{E} \left( \int_0^T |X_r^{t,x} - X_r^{t',x'}|^p dA_r^{t,x} \right) \\ & \quad \left. + \left( \mathbb{E} \sup_{0 \leq s \leq T} |A_s^{t,x} - A_s^{t',x'}|^p \right)^{1/2} \right]. \end{aligned}$$

Next, using Proposition 3.2 one can derive

$$\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p \right) \leq C(|t - t'|^{p/2} + |x - x'|^p + |t - t'|^{p/4} + |x - x'|^{p/2}).$$

Therefore, it suffice to choose  $p = \gamma$  conveniently to get

$$\mathbb{E} \left( \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^\gamma \right) \leq C(|t - t'|^{1+\beta} + |x - x'|^{d+\delta}).$$

We conclude from the last estimate, using Kolmogorov's lemma, that  $\{Y_s^{t,x}, s, t \in [0, T], x \in \overline{\Theta}\}$  has an a.s. continuous version.  $\square$

Now, for each  $(t, x) \in [0, T] \times \overline{\Theta}$ ,  $n \geq 1$ , we consider the following BDSDE,

$$\begin{aligned} {}^n Y_s^{t,x} &= l(X_T^{t,x}) + \int_s^T f_n(r, X_r^{t,x}, {}^n Y_r^{t,x}, {}^n Z_r^{t,x}) dr \\ &\quad + \int_s^T \phi(r, X_r^{t,x}, {}^n Y_r^{t,x}) dA_r^{t,x} + \int_s^T g(r, X_r^{t,x}, {}^n Y_r^{t,x}) \overleftarrow{dB}_r - \int_s^T {}^n Z_r^{t,x} dW_r, \end{aligned} \quad (3.15)$$

where  $f_n(t, x, y, z) = f(t, x, y, z) + n(y - h(t, x))^-$ .

Let  $\{{}^n Y_s^{t,x}, {}^n Z_s^{t,x}, t \leq s \leq T\}$  denotes the solution of BDSDE (3.15) and define  $u^n(t, x) = {}^n Y_t^{t,x}$ . It is shown in Boufoussi et al. [1] that the function  $v^n(t, x) = \varepsilon(t, x, u^n(t, x))$  is an  $\omega$ -wise viscosity solution to the following SPDE

$$\left\{ \begin{array}{l} (i) \frac{\partial u^n(t, x)}{\partial t} + Lu^n(t, x) + \tilde{f}_n(t, x, u^n(t, x), \sigma^*(x) D_x u^n(t, x)) = 0, \quad (t, x) \in [0, T] \times \Theta \\ (ii) \frac{\partial u^n}{\partial n}(t, x) + \tilde{\phi}(t, x, u^n(t, x)) = 0, \quad (t, x) \in [0, T] \times \partial\Theta, \\ (iii) u^n(T, x) = l(x), \quad x \in \overline{\Theta}, \end{array} \right.$$

where  $\tilde{f}_n(t, x, y, z) = \tilde{f}(t, x, y, z) + \frac{1}{D_y \eta(t, x, y)} n(y - \tilde{h}(t, x))^-$ .

Let us define for  $(t, x) \in [0, T]$ ,  $u(t, \omega, x) = Y_t^{t,x}$  and  $v(t, x) = \varepsilon(t, x, u(t, x))$ . It follows from penalization argument, that (along a subsequence)

$$|v^n(\tau, \xi) - v(\tau, \xi)| \rightarrow 0, \quad a.s.$$

as  $n$  goes to infinity.

Our main result in this section is the following:

**Theorem 3.1.** *Let assumptions (A1)-(A5) be satisfied. Then the function  $u(t, x)$  defined above is a stochastic viscosity solution of obstacle problem  $\mathcal{O}\mathcal{P}^{(f, \phi, g, h, l)}$ .*

*Proof.* By the definition of  $u$  it is easy to see that  $u(T, x) = l(x)$ . Now, since  $Y_s^{t,x}$  is  $\mathcal{F}_{t,s}^W \otimes \mathcal{F}_{s,T}^B$ -measurable, it follows that  $Y_t^{t,x}$  is  $\mathcal{F}_{t,T}^B$ -measurable. Consequently,  $u(t, x)$  is  $\mathcal{F}_{t,T}^B$ -measurable and so it is independent of  $\omega' \in \Omega'$ . Therefore, combining this result with Proposition 3.3, we obtain  $u \in C(\mathbf{F}^B; [0, T] \times \overline{\Theta})$ . On the other hand, it follows from its definition that for all  $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^0(\mathcal{F}_\tau^B; \overline{\Theta})$ ,

$$u(\tau(\omega), \xi(\omega)) = Y_{\tau(\omega)}^{\tau(\omega), \xi(\omega)} \geq h(\tau(\omega), \xi(\omega)), \quad \mathbb{P}\text{-a.s.}$$

Thus it remains to show that  $u$  satisfies (3.2)-(3.3) and (3.4)-(3.5). Using Proposition 3.1, it suffices to prove that  $v$  satisfies (3.9) and (3.10). To this end, let  $\omega \in \Omega$  be fixed such that

$$|v^n(\omega, t, x) - v(\omega, t, x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.16)$$

and consider  $(\tau, \xi, \varphi) \in \mathcal{M}_{0,T}^B \times L^0(\mathcal{F}_\tau^B; \overline{\Theta}) \times C^{1,2}(\mathcal{F}_\tau^B, [0, T] \times \overline{\Theta})$  verify, for such fixed  $\omega$ ,  $0 < \tau(\omega) < T$ ,  $v(\omega, \tau(\omega), \xi(\omega)) > \tilde{h}(\omega, \tau(\omega), \xi(\omega))$  and the inequality

$$v(\omega, \tau, x) - \varphi(\omega, \tau, x) < 0 = v(\omega, \tau(\omega), \xi(\omega)) - \varphi(\omega, \tau(\omega), \xi(\omega)) \quad (3.17)$$

for all  $(t, x)$  in some neighborhood  $\mathcal{V}(\omega, \tau(\omega), \xi(\omega))$  of  $(\tau(\omega), \xi(\omega))$ . Then there exists sequence  $(\tau_n(\omega), \xi_n(\omega), \varphi_n(\omega))_{n \geq 1} \in [0, T] \times \bar{\Theta} \times C^{1,2}([0, T] \times \bar{\Theta})$  satisfy

$$\begin{aligned}\tau_n(\omega) &\rightarrow \tau(\omega), \\ \xi_n(\omega) &\rightarrow \xi(\omega), \\ \varphi_n(\omega) &\rightarrow \varphi(\omega),\end{aligned}$$

such that the inequality

$$v^n(\omega, t, x) - \varphi_n(\omega, t, x) < 0 = v^n(\omega, \tau_n(\omega), \xi_n(\omega)) - \varphi_n(\omega, \tau_n(\omega), \xi_n(\omega))$$

holds for all  $(t, x)$  in some neighborhood  $\mathcal{V}(\tau_n(\omega), \xi_n(\omega)) \subset \mathcal{V}(\tau(\omega), \xi(\omega))$  and a suitable subsequence of  $(v^n)_{n \geq 1}$ . Using the fact that  $v^n(\omega, \cdot, \cdot)$  is a (deterministic) viscosity solution of the PDE  $(\tilde{f}_n(\omega, \cdot), \tilde{\phi}(\omega, \cdot), 0, l)$ , we obtain:

(a) if  $\xi_n(\omega) \in \Theta$  the inequality

$$A_{\tilde{f}_n, 0}(\varphi_n(\omega, \tau_n(\omega), \xi_n(\omega))) - D_t \varphi_n(\omega, \tau_n(\omega), \xi_n(\omega)) \leq 0$$

holds;

(b) if  $\xi_n(\omega) \in \partial\Theta$ , the inequality

$$\begin{aligned}\min \left[ A_{\tilde{f}_n, 0}(\varphi_n(\omega, \tau_n(\omega), \xi_n(\omega))) - D_t \varphi_n(\omega, \tau_n(\omega), \xi_n(\omega)), \right. \\ \left. - \frac{\partial \varphi_n}{\partial n}(\omega, \tau_n(\omega), \xi_n(\omega)) - \tilde{\phi}(\omega, \tau_n(\omega), \xi_n(\omega), \Psi_n(\omega, \tau_n(\omega), \xi_n(\omega))) \right] \leq 0\end{aligned}$$

holds.

On the other hand, since  $v(\omega, \tau(\omega), \xi(\omega)) > \tilde{h}(\omega, \tau(\omega), \xi(\omega))$ , it follows from (3.16) that  $v^n(\omega, \tau(\omega), \xi(\omega)) > \tilde{h}(\omega, \tau(\omega), \xi(\omega))$  for  $n$  large enough such that passing to the limit in the two last inequalities, we get:

(a) if  $\xi(\omega) \in \Theta$ , the inequality

$$A_{\tilde{f}, 0}(\varphi(\omega, \tau(\omega), \xi(\omega))) - D_t \varphi(\omega, \tau(\omega), \xi(\omega)) \leq 0$$

holds;

(b) if  $\xi(\omega) \in \partial\Theta$ , the inequality

$$\begin{aligned}\min \left[ A_{\tilde{f}, 0}(\varphi(\omega, \tau(\omega), \xi(\omega))) - D_t \varphi(\omega, \tau(\omega), \xi(\omega)), \right. \\ \left. - \frac{\partial \varphi}{\partial n}(\omega, \tau(\omega), \xi(\omega)) - \tilde{\phi}(\omega, \tau(\omega), \xi(\omega), \Psi(\omega, \tau(\omega), \xi(\omega))) \right] \leq 0\end{aligned}$$

holds.

□

### Acknowledgments

This work is partially done when the first author was post doctoral internship at Cadi Ayyad University of Marrakech. He would like to express his deep gratitude to B. Boufoussi, Y. Ouknine and UCAM Mathematics Department for their friendly hospitality. An anonymous referee is also acknowledged for his comments, remarks and for a significant improvement to the overall presentation of this paper.

## References

- [1] Boufoussi, B.; van Casteren, J.; Mrhardy, N. Generalized Backward doubly stochastic differential equations and SPDEs with nonlinear Neumann boundary conditions. *Bernoulli* **13** (2007), no.2, 423 – 446.
- [2] Buckdahn R.; Ma J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. Part I, *Stochastic process. Appl.* **93** (2001), no.2, 181 – 204.
- [3] Buckdahn R.; Ma J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. Part II, *Stochastic process. Appl.* **93** (2001), no.2, 205 – 228.
- [4] Buckdahn R.; Ma J. Pathwise stochastic Taylor expansions and stochastic viscosity solutions for fully nonlinear stochastic PDEs. *Ann. Appl. Probab.* **30** (2002), no.3, 1131 – 1171.
- [5] Crandall M.; Ishii H.; Lions P.L. User's guide to the viscosity solutions of second order partial differential equations. *Bull. A.M.S.* **27** (1992), no.1, 1 – 67.
- [6] Dellacherie, C.; Meyer, P. Probabilities and Potential. North-Holland Mathematics Studies, **29**. North-Holland Publishing Co., Amsterdam-New York; North-Holland Publishing Co., Amsterdam-New York, 1978. viii+189 pp. North Holland, (1978)
- [7] El Karoui, N.; Kapoudjian, C.; Pardoux, E.; Peng, S.; Quenz, M. C. Reflected solution of backward SDE's, and related obstacle problem for PDE's, *Ann. Probab.* **25** (1997), no.2, 702 – 737.
- [8] Lions P-L.; Souganidis P.E. Fully nonlinear stochastic partial differential equations. *C.R. Acad. Sc. Paris, Sér. I Math.* **326** (1998), no.1, 1085 – 1092.
- [9] Lions, P.L.; Sznitman, A.S. Stochastic differential equation with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37**, (1984), no. 4, 511 – 537
- [10] Matoussi, A., Stoica, L. The obstacle problem for quasilinear stochastic PDE's. *Ann. Probab.* **38** (2010), no. 3, 1143 – 1179.
- [11] Mohamed El Otmani, Reflected BSDE Driven by a Lévy Process. *J. Theoret. Probab.* **22** (2009), no. 3, 601 – 619.
- [12] Pardoux E.; Peng S. Adapted solution of backward stochastic differential equation, *Systems Control Lett.* **4** (1990), no.1, 55 – 61.
- [13] Pardoux, E.; Peng, S. Backward doubly stochastic differential equations and systems of quasilinear SPDEs. *Probab. Theory Related Fields* **98**, (1994), no.2, 209 – 227.
- [14] Pardoux, E.; Zhang, S. Generalized BSDEs and nonlinear Neumann boundary value problems, *Probab. Theory Related Fields* **110** (1998), no.4, 535 – 558.
- [15] Ren, Y.; Xia, N. Generalized reflected BSDE and obstacle problem for PDE with nonlinear Neumann boundary condition. *Stoch. Anal. Appl.* **24** (2006), no.5, 1013 – 1033, .



- [16] Słomiński, L. Euler's approximations of solutions of SDEs with reflecting boundary. *Stochastic process. Appl.* **94**, (2001), 317 – 337.
- [17] Yufen S.; Yanling G.; Kai L. Comparison theorem of backward doubly stochastic differential equations and application. *Stoch. Anal. Appl.* **23** (2005), no.1, 97 – 110.